Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 1, pp. 117–123, 2005

SINGULAR SOLUTIONS FOR AN ANISOTROPIC PLATE WITH AN ELLIPTICAL HOLE

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UDC 539.3

A solution of the bending problem for a plate with an elliptical hole subjected to a point force (a singular solution) is obtained using the engineering theory of thin anisotropic plates and Lekhnitskii's complex potentials. The solution is constructed by conformal mapping of the exterior of the elliptical hole onto the exterior of a unit circle with evaluation of the Cauchy-type integrals over closed contours. Different versions of the boundary conditions on the holw contour are considered. In the limiting case where the ellipse becomes a slot, the solution describes the bending of a plate with a rectilinear crack or a rigid inclusion.

Key words: *bending, anisotropic and isotropic materials, conformal mapping, Cauchy-type integral, unit circle.*

Let a point bending moment $m^* = m_x + im_y$ be applied at the point with coordinates $\tau = \xi + i\eta$ in an infinite anisotropic plate with an elliptical hole Λ (Fig. 1). Boundary conditions for the bending moments and transverse shear force (static conditions) or deflections and slopes (kinematic conditions) are specified on the hole contour. The solution of this problem reduces to constructing two analytic functions $F_\nu(z_\nu)$ governing the stress–strain state of the plate, where $z_\nu = x + \mu_\nu y$ are generalized complex coordinates ($\nu = 1, 2$) and μ_ν are roots of the characteristic equation ($\mu_1 \neq \mu_2$) [1].

The static boundary conditions are written in complex form as follows [1]:

$$2\,\mathrm{Re}\sum_{\nu=1}^{2}\frac{p_\nu}{\mu_\nu}\,\varphi_\nu(t_\nu) = -\int_0^{s(t)}(m\,dy + f\,dx) - Cx + C_1,$$

$$2\,\mathrm{Re}\sum_{\nu=1}^{2} q_\nu\varphi_\nu(t_\nu) = \int_0^{s(t)}(-m\,dx + f\,dy) + Cy + C_2, \qquad t \in \Lambda, \tag{1}$$

$$\varphi_\nu(t_\nu) = F'_\nu(t_\nu), \qquad f(s) = \int_0^{s(t)} p(s_0)\,ds_0.$$

Here $m(s)$ and $p(s)$ are the normal bending moments and transverse shear forces distributed along the contour and C, C_1, and C_2 are unknown real constants. Integration is performed along the arc of the contour from the starting point to the current point. Below, the contour is assumed to be traction-free [$m(s) = 0$ and $p(s) = 0$].

For a plate subjected to a point bending moment applied at an internal point, the complex potentials are given by

$$\varphi_\nu(z_\nu, \tau_\nu) = B_\nu \ln(z_\nu - \tau_\nu) + \varphi_{\nu 0}(z_\nu, \tau_\nu).$$

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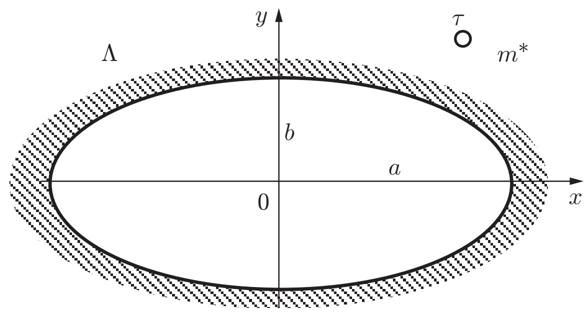

Fig. 1

Here $\varphi_{\nu 0}(z_\nu, \tau_\nu)$ are functions holomorphic in the exterior of the elliptical holes Λ_ν that correspond to Λ for the affine transformation $z_\nu = x + \mu_\nu y$ and B_ν are complex constants determined from the system

$$\sum_{\nu=1}^{2} (\mu_\nu^{k-2} B_\nu - \bar{\mu}_\nu^{k-2} \bar{B}_\nu) = f_k \qquad (k = \overline{1,4}\,),$$

$$f_1 = -\frac{m_y}{2\pi i D_{11}}, \qquad f_4 = -\frac{m_x}{2\pi i D_{22}}, \qquad f_j = 0 \quad (j = \overline{2,3}\,).$$

Here the constants D_{mn} are the flexural rigidities of the plate.

We multiply the first relation in the boundary conditions (1) by $-q_{3-\nu}$ and the second relation by $p_{3-\nu}/\mu_{3-\nu}$. Summing the resulting relations and collecting terms containing $\varphi_1(t_1)$, $\overline{\varphi_1(t_1)}$, $\varphi_2(t_2)$, and $\overline{\varphi_2(t_2)}$, we write boundary conditions (1) as

$$\varphi_\nu(t_\nu) - l_\nu \overline{\varphi_1(t_1)} - n_\nu \overline{\varphi_2(t_2)} = f_\nu^*(t),$$

$$f_\nu^*(t) = \Big(q_\nu \frac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu} \frac{p_\nu}{\mu_\nu}\Big)^{-1} \Big(-q_{3-\nu} f_1(t) + \frac{p_{3-\nu}}{\mu_{3-\nu}} f_2(t)\Big), \qquad t \in \Lambda, \tag{2}$$

$$f_1(t) = -\int_0^{s(t)} (m\,dy + f(t)\,dx) - Cx + C_1, \qquad f_2(t) = \int_0^{s(t)} (-m\,dx + f(t)\,dy) + Cy + C_2,$$

$$l_\nu = \frac{q_{3-\nu} \dfrac{\bar{p}_1}{\bar{\mu}_1} - \bar{q}_1 \dfrac{p_{3-\nu}}{\mu_{3-\nu}}}{q_\nu \dfrac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu} \dfrac{p_\nu}{\mu_\nu}}, \qquad n_\nu = \frac{q_{3-\nu} \dfrac{\bar{p}_2}{\bar{\mu}_2} - \bar{q}_2 \dfrac{p_{3-\nu}}{\mu_{3-\nu}}}{q_\nu \dfrac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu} \dfrac{p_\nu}{\mu_\nu}}.$$

Using the conformal mapping of the exterior of the unit circle $\gamma = \{|\sigma| = 1\}$ onto the exterior of the elliptical holes in the planes $z_\nu = x + \mu_\nu y$

$$z_\nu = \frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{a + i\mu_\nu b}{2}\,\frac{1}{\zeta_\nu} = \omega_\nu(\zeta_\nu), \qquad |\zeta_\nu| > 1$$

and the inverse functions

$$\zeta_\nu = \zeta_\nu(z_\nu) = \big(z_\nu + \sqrt{z_\nu^2 - (a^2 + \mu_\nu^2 b^2)}\,\big)/(a - i\mu_\nu b)$$

and introducing the notation $\varphi_\nu^*(\zeta_\nu, \eta_\nu) = \varphi_\nu(z_\nu, \tau_\nu)$ and $\eta_\nu = \zeta_\nu(\tau_\nu)$, we obtain the following expressions for the boundary values of the functions $\varphi_\nu^*(\zeta_\nu, \eta_\nu)$ on γ:

$$\begin{gathered} \varphi_\nu^*(\sigma, \eta_\nu) - l_\nu \overline{\varphi_1^*(\sigma, \eta_1)} - n_\nu \overline{\varphi_2^*(\sigma, \eta_2)} = f_\nu^*(\sigma), \\ \varphi_\nu^*(\zeta_\nu, \eta_\nu) = B_\nu \ln(\zeta_\nu - \eta_\nu) + \varphi_{\nu 0}^*(\zeta_\nu, \eta_\nu). \end{gathered} \tag{3}$$

Here $\varphi_{\nu 0}^*(\zeta_\nu, \eta_\nu)$ are unknown functions analytic outside the unit circle γ.

We multiply both sides of the boundary condition (3) by $d\sigma/(\sigma-\zeta_\nu)$, where ζ_ν lies in the exterior of the unit circle, and evaluate the Cauchy-type integrals for the functions determined on the unit circle contour [2]. Relations (3) imply the following expressions for $\varphi^*_{\nu 0}(\zeta_\nu,\eta_\nu)$:

$$\varphi^*_{\nu 0}(\zeta_\nu,\eta_\nu) = l_\nu \bar{B}_1 \ln\frac{\zeta_\nu\bar{\eta}_1-1}{\zeta_\nu\bar{\eta}_1} + n_\nu\bar{B}_2 \ln\frac{\zeta_\nu\bar{\eta}_2-1}{\zeta_\nu\bar{\eta}_2} + \frac{CD_\nu}{\zeta_\nu},$$
$$D_\nu = \frac{1}{2}\Big(aq_{3-\nu} + ib\,\frac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\frac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\frac{p_\nu}{\mu_\nu}\Big)^{-1}. \tag{4}$$

To determine the unknown constant C in relation (4), one should employ the condition that the deflection is a single-valued function in circulation about the unit circle γ

$$2\,\mathrm{Re}\Big\{\sum_{\nu=1}^{2} F_\nu(z_\nu)\Big\}_L = 0.$$

This relation implies the following formula for the constant (for a traction-free hole)

$$C = \mathrm{Im}\sum_{\nu=1}^{2}\frac{a-i\mu_\nu b}{2}\Big(l_\nu\,\frac{\bar{B}_1}{\bar{\eta}_1} + n_\nu\,\frac{\bar{B}_2}{\bar{\eta}_2}\Big)\Big[\mathrm{Im}\sum_{\nu=1}^{2}\frac{a-i\mu_\nu b}{4}\,\frac{aq_{3-\nu}+ibp_{3-\nu}/\mu_{3-\nu}}{q_\nu p_{3-\nu}/\mu_{3-\nu} - q_{3-\nu}p_\nu/\mu_\nu}\Big]^{-1}.$$

The bending and twisting moments in the plate are given by [1]

$$(M_x, M_y, H_{xy}) = -2\,\mathrm{Re}\Big\{\sum_{\nu=1}^{2}(p_\nu, q_\nu, r_\nu)\Phi_\nu(z_\nu)\Big\},$$

$$\Phi_\nu(z_\nu) = \frac{d\varphi_\nu(\zeta_\nu,\eta_\nu)}{d\zeta_\nu}\,[\omega'_\nu(\zeta_\nu)]^{-1}.$$

The potentials $\Phi^*_\nu(\zeta_\nu,\eta_\nu) = \varphi^{*\prime}_\nu(\zeta_\nu,\eta_\nu)$ become

$$\Phi^*_\nu(\zeta_\nu,\eta_\nu) = \frac{B_\nu}{\zeta_\nu-\eta_\nu} + \frac{\bar{B}_1 l_\nu}{\zeta_\nu(\zeta_\nu\bar{\eta}_1-1)} + \frac{\bar{B}_2 n_\nu}{\zeta_\nu(\zeta_\nu\bar{\eta}_2-1)} - \frac{CD_\nu}{\zeta_\nu^2}. \tag{5}$$

The kinematic conditions on the elliptical-hole contour are written as [1]

$$2\,\mathrm{Re}\sum_{\nu=1}^{2}F_\nu(t_\nu) = w^*(s), \qquad 2\,\mathrm{Re}\sum_{\nu=1}^{2}\varphi_\nu(t_\nu)(\mu_\nu\sin\vartheta+\cos\vartheta) = \frac{\partial w^*}{\partial n} \quad (t\in\Lambda)$$

and can be reduced, using the method described above (differentiation of the first condition with respect to the arc length s) to relations of the form (2), in which one should set

$$f^*_\nu(t) = \frac{f_2(t)-\mu_{3-\nu}f_1(t)}{\mu_\nu-\mu_{3-\nu}}, \qquad l_\nu = \frac{\mu_{3-\nu}-\bar{\mu}_1}{\mu_\nu-\mu_{3-\nu}}, \qquad n_\nu = \frac{\mu_{3-\nu}-\bar{\mu}_2}{\mu_\nu-\mu_{3-\nu}},$$

$$f_1(t) = -\frac{\partial w^*}{\partial s}\,\sin\vartheta + \frac{\partial w^*}{\partial n}\,\cos\vartheta, \qquad f_2(t) = \frac{\partial w^*}{\partial s}\,\cos\vartheta + \frac{\partial w^*}{\partial n}\,\sin\vartheta.$$

Here w^* and $\partial w^*/\partial n$ are the deflections and slopes specified at the edge of the elliptical hole, respectively, and ϑ is the angle between the normal to the hole contour and the x axis. If the deflections and slopes on the hole contour vanish, the boundary condition becomes homogeneous. The expressions for the complex potentials [2] can be obtained in a similar way to Eqs. (4). In the case of a point moment, the functions are expressed as $\varphi^*_{\nu 0}(\zeta_\nu,\eta_\nu)$ as

$$\varphi^*_{\nu 0}(\zeta_\nu,\eta_\nu) = l_\nu\bar{B}_1\ln\frac{\zeta_\nu\bar{\eta}_1-1}{\zeta_\nu\bar{\eta}_1} + n_\nu\bar{B}_2\ln\frac{\zeta_\nu\bar{\eta}_2-1}{\zeta_\nu\bar{\eta}_2}.$$

In the problem considered, we obtain the following formula for the complex potentials $\Phi^*_\nu(\zeta_\nu,\eta_\nu)$:

$$\Phi^*_\nu(\zeta_\nu,\eta_\nu) = \frac{B_\nu}{\zeta_\nu-\eta_\nu} + \frac{\bar{B}_1 l_\nu}{\zeta_\nu(\zeta_\nu\bar{\eta}_1-1)} + \frac{\bar{B}_2 n_\nu}{\zeta_\nu(\zeta_\nu\bar{\eta}_2-1)}.$$

The last two formulas can be obtained from (4) and (5) by setting $C=0$.

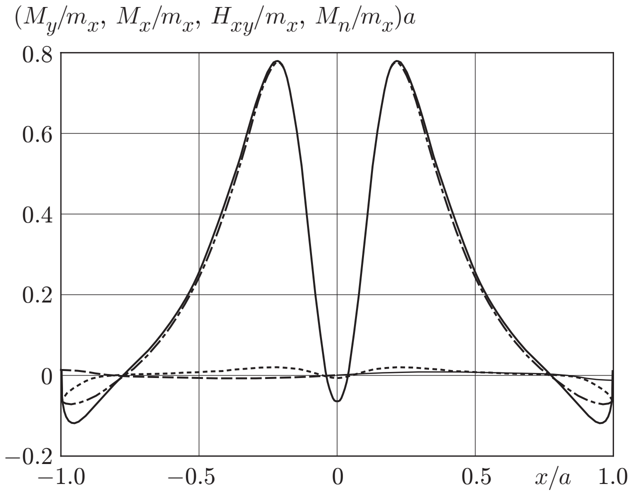

Fig. 2

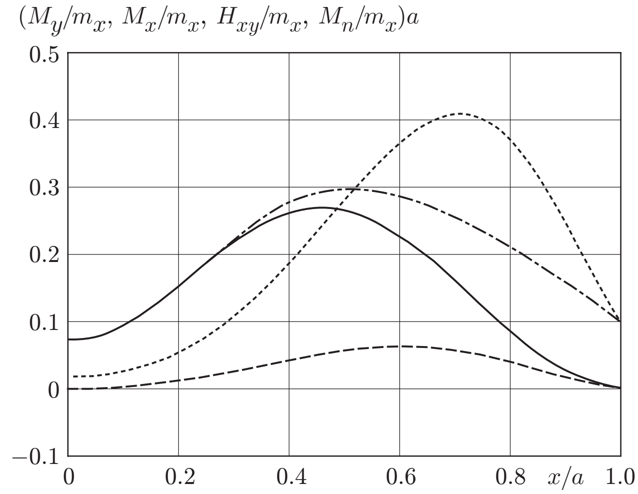

Fig. 3

TABLE 1

Material number.	$E_1 \cdot 10^{-4}$, MPa	$E_2 \cdot 10^{-4}$, MPa	E_1/E_2	ν_1	$G \cdot 10^{-4}$, MPa
1	27.610	27.610	1	0.25	11.044
2	27.610	2.761	10	0.25	1.035

It is not possible to solve the plate problem subject to mixed boundary conditions on the hole contour (for example, a simply supported contour) using the method described above because the boundary conditions for the function $\varphi_\nu^*(\zeta_\nu, \eta_\nu)$ on the unit circle contour γ cannot be written in the form of (3).

We consider some numerical results obtained by the formulas given above. Figure 2 shows the distribution of the bending and twisting moments M_y (solid curve), M_x (dotted curve), H_{xy} (dashed curve), M_n (dot-and-dashed curve) along the clamped edge of an elliptical hole ($b/a = 0.5$) in a plate subjected to a point bending moment m_x at a point lying on the continuation of the minor axis of the ellipse. The coordinate of the load application point is $0.9a$, the plate material is boron–epoxy composite (material No. 2 in Table 1), and the angle between the principal anisotropy direction E_1 and the x axis is $\varphi = \pi/2$. Figure 3 shows the distribution of the bending and twisting moments M_y, M_x, H_{xy}, and M_n (notation same as in Fig. 2) in the boron–epoxy composite plate for $\varphi = 0$. The stress concentration on the plate contour is substantially reduced (by a factor of two) in this case. By virtue of symmetry, only one quarter of the ellipse is shown in Fig. 3.

Figure 4 shows the distribution of the bending and twisting moments M_y, M_x, H_{xy}, and M_n (notation same as in Fig. 2) along the hole contour of an isotropic plate (material No. 1 in Table 1). Here and below, the numerical results for the isotropic material ($\mu_{1,2} = \pm\alpha + i\beta$ for an orthotropic material with $\varphi = 0$) were obtained by passing to the limit as $\alpha \to 0$ and $\beta \to 1$. In fact, we used the value of one of the elastic moduli (for example, the shear modulus G) in the isotropic material that differ from the exact value by a few hundredths of a percent. Numerical analysis shows that the error of the results does not exceed the error in approximating the elastic characteristic of the material. For this approximation, the predicted stresses are accurate to five or six significant figures.

Figure 5 shows the distribution of the bending and twisting moments along the traction-free edge of an elliptical hole ($b/a = 0.5$) in a plate loaded by a point bending moment m_x applied to a point lying on the continuation of the minor axis of the ellipse. Here M_θ denotes the bending moments in areas normal to the hole contour (light solid curve and the remaining notation is same as in Fig. 2). The coordinate of the load application point is $0.9a$ and the plate material is isotropic. Figure 6 shows the stress distribution in the same plate loaded by the point bending moment m_y. In this problem for an orthotropic plate ($\varphi = 0, \pi/2$), the real constant C vanishes if the bent surface has a plane of symmetry (for example, if the point moment m_x acts on the continuation of the

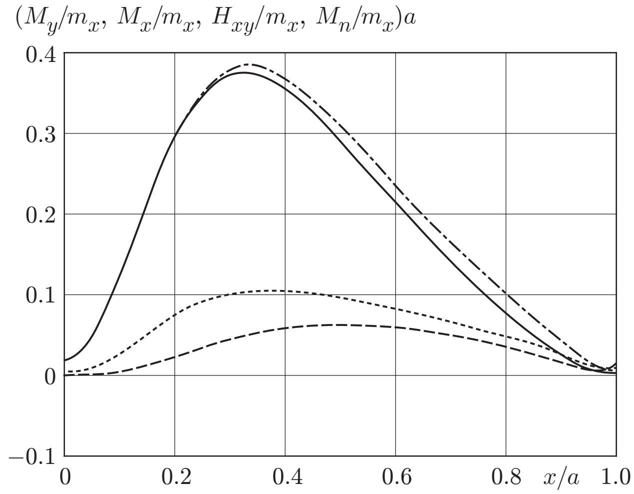

Fig. 4

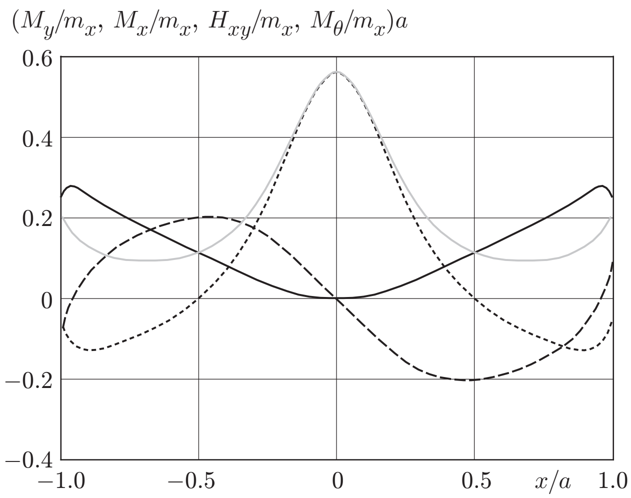

Fig. 5

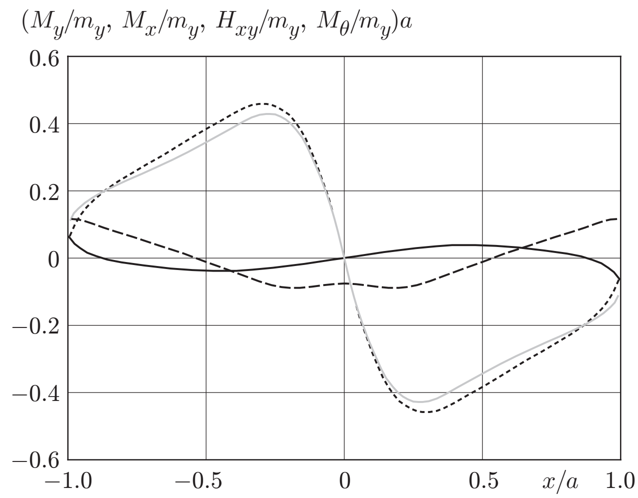

Fig. 6

minor axis of the ellipse). In the remaining cases, the constant is nonzero and depends on the type of load and its coordinate.

The technique described above can be used not only in the case of point loads. We consider the same plate with an elliptical hole clamped along the hole contour under the action of moments distributed uniformly at infinity. In this case, the stress–strain state (SSS) can be obtained by summing the SSS in the plate without a hole and the perturbed SSS due to the hole. For the plate without a hole, the complex potentials describing the homogeneous stress field are given by

$$\varphi_\nu(z_\nu) = B_\nu^* z_\nu + G_\nu,$$

where G_ν are arbitrary complex constants and the constants B_ν^* are determined from the system [1]

$$-2\,\mathrm{Re}\Big\{\sum_{\nu=1}^{2} p_\nu B_\nu^*\Big\} = M_{x0}, \qquad -2\,\mathrm{Re}\Big\{\sum_{\nu=1}^{2} q_\nu B_\nu^*\Big\} = M_{y0}, \quad -2\,\mathrm{Re}\Big\{\sum_{\nu=1}^{2} r_\nu B_\nu^*\Big\} = H_{xy0}, \qquad \mathrm{Re}\,\{B_1^*\} = 0.$$

Here M_{x0}, M_{y0}, and H_{xy0} are uniform loads at infinite distance from the hole.

The functions $\varphi_\nu^*(\zeta_\nu)$ describing the SSS of the plate with a hole become

$$\varphi_\nu^*(\zeta_\nu) = B_\nu^*\Big(\frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{a + i\mu_\nu b}{2}\,\frac{1}{\zeta_\nu}\Big) + \varphi_{\nu 0}^*(\zeta_\nu).$$

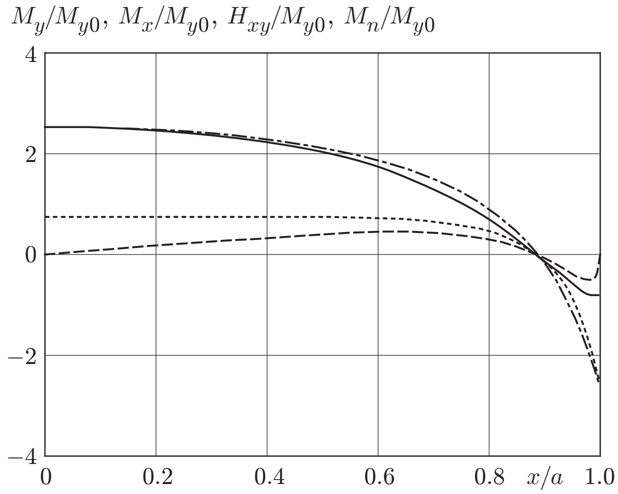

Fig. 7

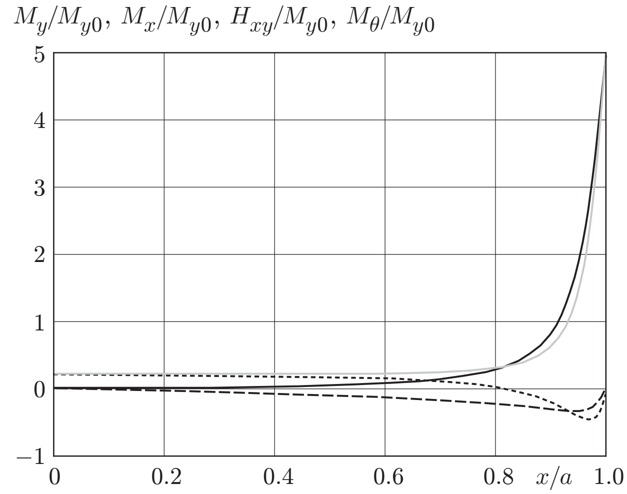

Fig. 8

Here $\varphi^*_{\nu 0}(\zeta_\nu)$ are unknown functions analytic outside the hole contour that describe the SSS perturbations induced by the hole. The expressions of the complex potentials are derived similarly to relations (4):

$$\varphi^*_\nu(\zeta_\nu) = B^*_\nu \,\frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{1}{\zeta_\nu}\Big[l_\nu \bar{B}^*_1\,\frac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}^*_2\,\frac{a + i\bar{\mu}_2 b}{2}\Big].$$

If stresses are specified on the hole contour, the right side of the boundary conditions (2) and, hence, (3) contain the unknown real constants C, C_1, and C_2 [see (1)]. The real constants C_1 and C_2 specify the rigid-body rotation of the plate and have no effect on the stresses. To determine the real constant C, it is necessary to use the additional condition — the deflection should be a single-valued function in circulation about the elliptic-hole contour. For the traction-free contour, this condition yields

$$C = -\frac{\mathrm{Im}\displaystyle\sum_{\nu=1}^{2}\frac{a - i\mu_\nu b}{2}\Big(- B^*_\nu\,\frac{a + i\mu_\nu b}{2} + l_\nu \bar{B}^*_1\,\frac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}^*_2\,\frac{a + i\bar{\mu}_2 b}{2}\Big)}{\mathrm{Im}\displaystyle\sum_{\nu=1}^{2}\frac{a - i\mu_\nu b}{4}\Big(aq_{3-\nu} + ib\,\frac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\frac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\frac{p_\nu}{\mu_\nu}\Big)^{-1}}.$$

In this case, the complex potentials become

$$\begin{aligned}\varphi^*_\nu(\zeta_\nu) &= B^*_\nu \,\frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{1}{\zeta_\nu}\Big[l_\nu \bar{B}^*_1\,\frac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}^*_2\,\frac{a + i\bar{\mu}_2 b}{2} + CD_\nu\Big],\\ \Phi_\nu(z_\nu) &= \frac{1}{\omega'_\nu(\zeta_\nu)}\Big[B^*_\nu \frac{a - i\mu_\nu b}{2} - \frac{1}{\zeta^2_\nu}\Big(l_\nu \bar{B}^*_1\,\frac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}^*_2\,\frac{a + i\bar{\mu}_2 b}{2} + CD_\nu\Big)\Big].\end{aligned} \tag{6}$$

Figure 7 shows the distribution of the bending and twisting moments M_y, M_x, H_{xy}, and M_n (notation same as in Fig. 2) along the contour of the elliptical hole in the plate (in the areas coinciding with the contour line) for the case of a clamped hole edge ($w^* = 0$ and $\partial w^*/\partial n = 0$). The numerical solution was compared with the results of Lekhnitskii [1], who obtained closed-form solution for an orthotropic plate with a rigid circular inclusion. Excellent agreement between the solutions is observed.

Figure 8 shows the distribution of the moments M_y, M_x, H_{xy}, and M_θ (notation same as in Fig. 2) along the traction-free contour of an elliptical hole (the first boundary-value problem) in a boron–epoxy composite plate ($\varphi = \pi/2$). The maximum bending moment $M_{\theta\,\max} = 4.9120 M_{y0}$ occurs at the points $(\pm a, 0)$. The numerical results agree with the solution for an anisotropic plate with a traction-free circular hole [1].

Setting $b = 0$ in (6), we obtain the solution of the problem of an infinite plate with a rectilinear crack (rigid inclusion). The complex potentials $\Phi_\nu(z_\nu)$ become

$$\Phi_\nu(z_\nu) = \frac{1}{2\sqrt{z_\nu - a}\sqrt{z_\nu + a}}\Big[B_\nu^* J(z_\nu) - \frac{a^2}{J(z_\nu)}\Big(\bar{B}_\nu^* l_\nu + \bar{B}_2^* n_\nu + 2\,\frac{CD_\nu}{a}\Big)\Big], \qquad J(z_\nu) = z_\nu + \sqrt{z_\nu^2 - a^2}.$$

The last relation implies that in the vicinity of the crack tip (rigid inclusion), the stresses have a singularity of order $1/\sqrt{\rho}$, where ρ is the distance from the crack tip [3]. In a similar manner, one can obtain the solution of this problem for point loads [4]. In this case, the stress singularity at the crack tip has the same form: $1/\sqrt{\rho}$.

For the case of an elliptical hole loaded by constant bending moments at the edge of the hole and a load specified at infinity:

$$C = -\frac{\operatorname{Im}\sum\limits_{\nu=1}^{2}\dfrac{a - i\mu_\nu b}{2}\Big(-B_\nu^*\,\dfrac{a + i\mu_\nu b}{2} + l_\nu \bar{B}_1^*\,\dfrac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}_2^*\,\dfrac{a + i\bar{\mu}_2 b}{2}\Big)}{\operatorname{Im}\sum\limits_{\nu=1}^{2}\dfrac{a - i\mu_\nu b}{4}\Big(aq_{3-\nu} + ib\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\dfrac{p_\nu}{\mu_\nu}\Big)^{-1}}$$

$$- m\,\frac{\operatorname{Im}\sum\limits_{\nu=1}^{2}\dfrac{a - i\mu_\nu b}{4}\Big(ibq_{3-\nu} - a\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\dfrac{p_\nu}{\mu_\nu}\Big)^{-1}}{\operatorname{Im}\sum\limits_{\nu=1}^{2}\dfrac{a - i\mu_\nu b}{4}\Big(aq_{3-\nu} + ib\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\dfrac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\dfrac{p_\nu}{\mu_\nu}\Big)^{-1}}. \qquad (7)$$

In formula (7), m is the intensity of the constant normal moments applied to the hole contour (the distributed transverse load $p = 0$). In this case, the complex potentials become

$$\varphi_\nu^*(\zeta_\nu) = B_\nu^*\,\frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{1}{\zeta_\nu}\Big[l_\nu \bar{B}_1^*\,\frac{a + i\bar{\mu}_1 b}{2} + n_\nu \bar{B}_2^*\,\frac{a + i\bar{\mu}_2 b}{2} + CD_\nu + mE_\nu\Big],$$

$$E_\nu = \frac{1}{2}\Big(ibq_{3-\nu} - a\frac{p_{3-\nu}}{\mu_{3-\nu}}\Big)\Big(q_\nu\,\frac{p_{3-\nu}}{\mu_{3-\nu}} - q_{3-\nu}\,\frac{p_\nu}{\mu_\nu}\Big)^{-1}.$$

Calculation results show that for orthotropic plate materials ($\varphi = 0, \pi/2$), the constant C is independent of the semiaxes ratio of the ellipse, i.e., the value of this constant is the same for a circle and a rectilinear cut. The constant C has a significant effect on the satisfaction of the boundary conditions on the hole contour and, hence, on the stress–strain state of the plate.

Using (7), one can find the arbitrary constant in the bending problem for an infinite plate with a rectilinear crack whose edges are subjected to bending moments of constant intensity [5] (in this case, it is necessary to set $b = 0$ and $B_\nu^* = 0$). It is worth noting that in this problem for an orthotropic plate material ($\varphi = 0, \pi/2$), the constant C is identically equal to zero. If $\varphi \neq 0, \pi/2$, the constant is nonzero.

Based on the complex potentials obtained above, solutions can be constructed for point dislocations treated as displacement discontinuities in a plate [4]. These solutions can be used to construct complex potentials in bending problems for plates with an elliptical hole, through-thickness curvilinear cuts (cracks), and thin curvilinear rigid inclusions. In this case, the complex potentials are written as singular integrals containing an unknown density function determined from the conditions on the defect contour, which lead to a singular integral equation or a system of integral equations subject to additional relations.

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